Grader's Note

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1 Frequently Missed Problems

1.1 HW1. #3-(c)

Many students included the sentence like this: "to f^{-1} be defined, f should be bijective." You are absolutely right. However, as you can see at second quiz, f^{-1} can be defined, even if $f : X \to Y$ is not bijective, as a function from the collection of subsets of Y to the collection of subsets of X.

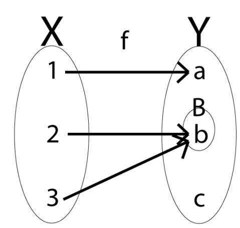

Figure 1: definition of X, Y and f

For example, let $X = \{1, 2, 3\}, Y = \{a, b, c\}$ and $B \subseteq Y$ such that $B = \{b\}$. Let a function $f : X \to Y$ as $f(1) = a, f(2) = b, f(3) = b$. You see f is not bijective, and $f^{-1} : Y \to X$ cannot be defined. Saying intuitively, if you put an element of Y into f^{-1}, then it returns more than one elements of X. That makes $f^{-1} : Y \to X$ is not a function!

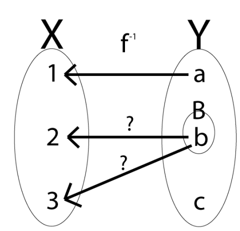

Figure 2: f^{-1} cannot be defined, as $f^{-1}(b)$ is not well defined, as a inverse function.

However, what if we put a subset of Y into f^{-1}? By considering definition of $f^{-1}(A)$, for any $A \subseteq Y$, $f^{-1}(A) = \{x \in X | f(x) = y \text{ for some } y \in Y\}$. $f^{-1}(A)$ is a subset of X. That is, we can intepret f^{-1} as a function from the collection of subsets of Y to the collection of subsets of X! You can check, if f^{-1} is bijective, this 'subset' definition still holds(but trivial.)

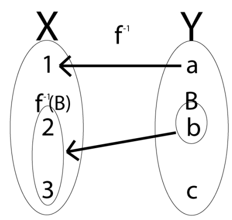

Figure 3: f^{-1} of subsets are well defined: just no problem!

In particular, for subsets of Y containing exactly one element, for example this case, $\{a\}, \{b\}, \{c\}$, *the fiber of* a, where $a \in Y$, is defined by $f^{-1}(\{a\})$. The fiber of a is a subset in X. The fiber of some useful element is used somewhere else, like algebra course, so it'll be better if you remember.

Of course, one thing you should remember in this course is this: for $f : X \to Y$ and any $A \subseteq Y$, $f^{-1}(A)$ is just a subset of X, and do not need the existence of an inverse function f^{-1}.

2 Comments

2.1 Very Easy $\epsilon - \delta$

$\epsilon - \delta$ definition of limit is one challenging concept that freshmen and sophomores in mathematics. While grading, I found that many students are having difficulty understanding $\epsilon - \delta$. $\epsilon - \delta$ comes with very crucial parts of analysis; limits, continuity, convergence$\cdots$. The sooner you understand it, the better.

Let us start with the limit of a sequence. Let $\{x_n : n \in \mathbb{N}\}$ be a real sequence. (x_n) converges to L if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - L| < \epsilon$. In this case we have N, instead of δ, but it's OK. You should understand the difference, however underlying idea is same. I think propessor and teaching assistants have taught you enough what the statement means. I'll just briefly check how you should use it when you prove something.

Suppose you are going to show $x_n \to L$ as $n \to \infty$. Then what you should show is this: for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n \geq N$. So whatever you do prove; that should holds for *all positive* ϵ. This can be satisfied by starting proof with those statement: "Given any $\epsilon > 0$, ..." or "Fix $\epsilon > 0$. ...", and then never ever attempt to overwrite ϵ until your proof is done. The next thing you do is to show the existence of N(or, δ, when you are dealing with continuous something). Whoa, showing existence sounds difficult. However, if you can find N, then it exists! Therefore, in many case, you should find N. Some instructor use notation $N(\epsilon)$ to emphasize N should be only dependent on ϵ, not depend on any other tricky indices you use, like n, k, at cetera. In this viewpoint, N is a function of ϵ. How you find the function $N(\epsilon)$? That is the point of the proof. That means, $N(\epsilon)$ should be decided upon the information of $\{x_n\}$. Given the information of sequence $\{x_n\}$, you should construct concrete $N(\epsilon)$, then your proof is done and flawless. Let's see some example.

Example 1 *Let $\{y_n\}$ and $\{z_n\}$ are two real sequences converge to L. Let's define $\{x_n\}$ with infinite coin toss:*

$$x_n = \begin{cases} y_n & \text{if } n\text{th coin toss shows HEAD} \\ z_n & \text{if } n\text{th coin toss shows TAIL} \end{cases}$$

In other way, put a random sequence of 0 and 1, for example, 0010110... then change 0 to entries of y_n sequencially, and 1 to z_n. in case 0010110..., $\{x_n\}$ becomes $\{y_1, y_2, z_3, y_4, z_5, z_6, y_7, ...\}$. Show $\{x_n\}$ converges to L.

You should show that, for every $\epsilon > 0$, there exists a natual number N such that $|x_n - L| < \epsilon$ for all $n \geq N$. You have the information of $\{x_n\}$; $\{x_n\}$ consists of two partitioning subsequences $\{y_n\}$ and $\{z_n\}$, and they are convergent to L. You should start here. Fix ϵ! Then what you can say about $\{x_n\}$ with ϵ? Well, We know $\{y_n\}$ and $\{z_n\}$ are both convergent, so there exist a N_y and N_z such that $|y_n - L| < \epsilon$ for $n \geq N_y$ and $|z_n - L| < \epsilon$ for $n \geq N_z$, respectively. Then, how can we define N so that we can sure "$n \geq N$ implies $|x_n - L| < \epsilon$"? Since $x_n = y_n$ or $x_n = z_n$ for any n,

$$|x_n - L| = |y_n - L| < \epsilon \text{ holds if } n \geq N_y \text{ and } x_n = y_n, \tag{1}$$
$$|x_n - L| = |z_n - L| < \epsilon \text{ holds if } n \geq N_z \text{ and } x_n = z_n. \tag{2}$$

Those statements are almost complete, but the additional conditions "$x_n = y_n$" and "$x_n = z_n$" is a nuisance. How can we get rid of them? Fortunatelly, we have either $x_n = y_n$ or $x_n = z_n$, two cases covers entire case. Then we should manipulate N_y and N_z part. You should understand

$$\text{“}|x_n - L| = |y_n - L| < \epsilon \text{ holds if } n \geq N_y \text{ and } x_n = y_n'' \text{ implies}$$
$$\text{“}|x_n - L| = |y_n - L| < \epsilon \text{ holds if } n \geq N^* \text{ and } x_n = y_n'' \text{ for any } N^* \geq N_y.$$

since this is just deleting some information you have. This holds for z_n part too. We want to unify two statements, (1) and (2), we are to find some number that is greater or equals to both N_y and N_z. There indeed is; $\max(N_y, N_z)$. Then from (1) and (2), we get

$$|x_n - L| < \epsilon \text{ holds if } n \geq \max(N_y, N_z) \text{ and } x_n = y_n, \tag{3}$$
$$|x_n - L| < \epsilon \text{ holds if } n \geq \max(N_y, N_z) \text{ and } x_n = z_n. \tag{4}$$

Then from (3) and (4),

$$|x_n - L| < \epsilon \text{ holds if } n \geq \max(N_y, N_z) \text{ and either } x_n = y_n \text{ or } x_n = z_n. \tag{5}$$

We always have $x_n = y_n$ or $x_n = z_n$, finally we have

$$|x_n - L| < \epsilon \text{ holds if } n \geq \max(N_y, N_z). \tag{6}$$

Then $\max(N_y, N_z)$ is what we are looking for: N. Let $N = \max(N_y, N_z)$, and verify this indeed holds. Ah, one more thing: N_y and N_z are functions of ϵ, therefore N is a function of ϵ too.

You can find this kind of examples everywhere, so please read them carefully to have a thorough understanding of $\epsilon - \delta$.

2.2 Inequality

Almost every time, you are asked to prove an equality. Equality is nice. Clear, Elegant, and easy to understand. However, when you are trying to prove an equality, you may find it is difficult to write a 'magic sentences of equalities'. You may have faced some equalities that is not easy to prove.

There is a popular notion about this situation: *inequality for analysis, equality for algebra*. Although both of analysis and algebra use both of inequality and equality, at least it is true that understanding how inequalities are used will get you a higher analysis grade. :)

Logic is simple. Let a, b are real numbers. If you have $a \geq b$ and $a \leq b$, then what you will get? It's simple: $a = b$! Therefore, if you are asked to show $a = b$, then it is suffice to show $a \geq b$ and $a \leq b$. That's simple. (Compare this statement with problem 6 of HW1.)

What really fascinating is, this kind of approach holds for other general inequalities also! Let A, B are two sets. We can show $A = B$ by showing $A \subseteq B$ (Technically, you can show this by every elements of A is also in B.) and $A \supseteq B$ hold. Sometimes, usually in number theory, you may show two natural numbers coincide ($x = y$) by showing they are dividing each other. ($x \mid y, y \mid x$) If you are interested what are other inequalities, just google 'partial order'.

One very crucial using of this "two inequalities $\rightarrow$ an equality" trick is proving equivalence: that is, showing $A \Leftrightarrow B$, when A and B are some statements. Showing two statements are equivalent, is equivalent to show each statement implies each other: $A \Rightarrow B$ and $A \Leftarrow B$. You may check problem 5 of HW1.

Example 2 *I mentioned that*

To show $A \Leftrightarrow B$ is equivalent to show $A \Rightarrow B$ and $A \Leftarrow B$.

How you are going to prove this statement? :) I'm not asking you to be detail. What's your idea?

Cheer up and work hard!